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On the construction of recurrence relations for the expansion and connection coefficients in series of Jacobi polynomials

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Abstract
Formulae expressing explicitly the Jacobi coefficients of a general-order derivative (integral) of an infinitely differentiable function in terms of its original expansion coefficients, and formulae for the derivatives (integrals) of Jacobi polynomials in terms of Jacobi polynomials themselves are stated. A formula for the Jacobi coefficients of the moments of one single Jacobi polynomial of certain degree is proved. Another formula for the Jacobi coefficients of the moments of a general-order derivative of an infinitely differentiable function in terms of its original expanded coefficients is also given. A simple approach in order to construct and solve recursively for the connection coefficients between Jacobi–Jacobi polynomials is described. Explicit formulae for these coefficients between ultraspherical and Jacobi polynomials are deduced, of which the Chebyshev polynomials of the first and second kinds and Legendre polynomials are important special cases. Two analytical formulae for the connection coefficients between Laguerre–Jacobi and Hermite–Jacobi are developed.

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1. Introduction

The main feature of spectral methods is to take various orthogonal systems of infinitely differentiable global functions as trial functions. Different trial functions lead to different spectral approximations; for instance, trigonometric polynomials for periodic problems, Chebyshev, Legendre, ultraspherical and Jacobi polynomials for non-periodic problems, Laguerre polynomials for problems on the half line, and Hermite polynomials for problems on the whole line. The fascinating merit of spectral methods is their high accuracy, the so-called convergence of 'infinite order'.

Classical orthogonal polynomials are used successfully and extensively for the numerical solution of differential equations in spectral and pseudospectral methods (see, for instance, Ben-Yu (1998a, 1998b), Coutsias *et al* (1996), Doha (1990, 2000), Doha and Helal (1997), Doha and Al-Kholi (2001), Doha and Abd-Elhameed (2002), Haidvogel and Zang (1979) and Siyyam and Syam (1997)). In particular, Lewanowicz (1986, 1991, 1992) has developed three different algorithms for constructing recurrence relations for the expansion coefficients in Jacobi series solutions for linear ordinary differential equations with polynomial coefficients. Solutions of such recurrence relations enable one to obtain the spectral approximations in Jacobi series expansions for the differential equations under consideration.

It is well known (Canuto *et al* 1998) that the eigenfunctions of certain singular Sturm–Liouville problems allow the approximation of function in $C^{\infty}[a, b]$ whose truncation error approaches zero faster than any finite negative power of the number of basis functions (retained modes) used in the approximation, as the number (order of truncation N) tends to infinity. The importance of Sturm–Liouville problems for spectral methods lies in the fact that the spectral approximation of the solution of a differential equation is usually regarded as a finite expansion of eigenfunctions of a suitable Sturm–Liouville problem.

It is proved that the Jacobi polynomials are precisely the only polynomials arising as eigenfunctions of a singular Sturm–Liouville problem (cf Canuto *et al* (1998), section 9.2). This class of polynomials comprises all the polynomial solutions to singular Sturm–Liouville problems on $[-1, 1]$. Chebyshev, Legendre and ultraspherical polynomials are particular cases of the Jacobi polynomials. These polynomials have been used in both the solution of boundary value problems (Fox and Parker 1972, Gottlieb and Orszag 1997) and in computational fluid dynamics (Canuto *et al* 1998, Peyret 2002, Voigt *et al* 1984). In most of these applications use is made of formulae relating the expansion coefficients of derivatives appearing in the differential equation to those of the function itself. This process results in an algebraic system or a system of differential equations for the expansion coefficients of the solution which then must be solved.

Formulae for the expansion coefficients of a general-order derivative of an infinitely differentiable function in terms of those of the function are available for expansions in Chebyshev (Karageorghis 1988a), Legendre (Phillips 1988), ultraspherical (Karageorghis and Phillips 1989, 1992, Doha 1991), Jacobi (Doha 2002a), Hermite (Doha 2003c) and Laguerre (Doha 2003b) polynomials.

An alternative approach to differentiating solution expansions is to integrate the differential equation q times, where q is the order of the equation. An advantage of this approach is that the general equation in the algebraic system then contains a finite number of terms. Phillips and Karageorghis (1990) and Doha (2002b) have followed this approach to obtain a formula relating the expansion coefficients of an infinitely differentiable function that have been integrated an arbitrary number of times in terms of the expansion coefficients of the function when the expansion functions are ultraspherical polynomials. The corresponding formula when the expansion functions are Jacobi polynomials is given in Doha (2003a).

A more general situation which often arises in the numerical solution of differential equations with polynomial coefficients in spectral and pseudospectral methods is the evaluation of the expansion coefficients of the moments of high-order derivatives of infinitely differentiable functions. A formula for the shifted Chebyshev coefficients of the moments of a general-order derivative of an infinitely differentiable function is given in Karageorghis (1988b). Corresponding results for Chebyshev polynomials of the first and second kinds, Legendre, ultraspherical, Hermite and Laguerre polynomials are given in Doha (1994), Doha and El-Soubhy (1995) and Doha (1998, 2003c, 2003b) respectively.

Up to now, and to the best of our knowledge, many formulae corresponding to those mentioned previously are unknown and are traceless in the literature for the Jacobi expansions. This partially motivates our interest in such polynomials. Another motivation is that the theoretical and numerical analysis of numerous physical and mathematical problems very often requires the expansion of an arbitrary polynomial or the expansion of an arbitrary function with its derivatives and moments into a set of orthogonal polynomials. This is particularly true for Jacobi polynomials. To be precise, the Laguerre and Jacobi polynomials, which virtually cover all the classical orthogonal polynomials, play an important role in various physical applications. In many cases, the solutions of the Schrödinger equation for simple systems are expressed directly in terms of such polynomials; for example, hydrogen-like functions via Laguerre polynomials, rotator functions via the Jacobi polynomials, etc. Since the Hermite and Bessel polynomials are particular cases of the Laguerre polynomials, and the Chebyshev of the first and second kinds, the Legendre and ultraspherical polynomials are particular cases of the Jacobi polynomials, the numbers of such examples may be easily extended. The Laguerre and Jacobi polynomials also play an important role in approximate variational solutions of complex many-electron systems, because basis functions in variational methods are frequently connected with these two classes of special functions. This also motivates our interest in such polynomials. Another motivation is, once we give the formulae concerning the Jacobi polynomials, corresponding to those obtained by Doha (1994) for Chebyshev, Doha and El-Soubhy (1995) for Legendre, Doha (1998, 2002b) for ultraspherical, Doha (2003c) for Hermite and Doha (2003b) for Laguerre polynomials, the subject of classical continuous cases will be closed.

The paper is organized as follows. In section 2, we give some relevant properties of Jacobi polynomials, and state without proofs four theorems from Doha (2002a) and Doha (2003a). Two of them give explicitly two formulae for the coefficients of a general-order derivative (integral) of an expansion in Jacobi polynomials in terms of the coefficients of the original expansion; while the other two express explicitly formulae for the derivatives (integrals) of Jacobi polynomials of any degree and for any order in terms of the Jacobi polynomials themselves. In section 3, we prove a theorem which gives the Jacobi coefficients of the moments of one single Jacobi polynomial of any degree. Another theorem which expresses the Jacobi coefficients of the moments of a general-order derivative of an infinitely differentiable function in terms of its Jacobi coefficients is proved in section 4. Results for ultraspherical, Chebyshev of the first and second kinds and Legendre polynomials are deduced as particular cases of Jacobi polynomials. A simple approach in order to build and solve recursively for the connection coefficients between two different families of Jacobi orthogonal polynomials is described in section 5. Corresponding results for ultraspherical polynomials and their important special cases are noted. Formulae for the connection coefficients between Laguerre–Jacobi and Hermite–Jacobi are also developed. An application for solving ordinary differential equations with varying coefficients, by reducing them to recurrence relations in the expansion coefficients of the solution, is given in section 6.

2. Some properties of Jacobi polynomials

The Jacobi polynomials associated with the real parameters ($\alpha > -1, \beta > -1$) (see Szegö (1985)), are a sequence of polynomials $P_n^{(\alpha,\beta)}(x)$ ($n = 0, 1, 2, \ldots$), each respectively of degree n, satisfying the orthogonality relation

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)\,\mathrm{d}x = \begin{cases} 0 & m \neq n \\ h_n & m = n \end{cases}$$

where

$$h_n = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)}. \tag{1}$$

These polynomials are eigenfunctions of the following singular Sturm–Liouville equation:

$$(1-x^2)\phi''(x) + [\beta - \alpha - (\alpha+\beta+2)x]\phi'(x) + n(n+\alpha+\beta+1)\phi(x) = 0.$$

A consequence of this is that spectral accuracy can be achieved for expansions in Jacobi polynomials. For our present purposes it is convenient to standardize the Jacobi polynomials so that

$$P_n^{(\alpha,\beta)}(1) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} \qquad P_n^{(\alpha,\beta)}(-1) = \frac{(-1)^n\Gamma(n+\beta+1)}{n!\Gamma(\beta+1)}.$$

In this form the polynomials may be generated using the standard recurrence relation of Jacobi polynomials starting from $P_0^{(\alpha,\beta)}(x) = 1$ and $P_1^{(\alpha,\beta)}(x) = \frac{1}{2}[\alpha - \beta + (\lambda+1)x]$, or obtained from Rodrigue's formula

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta} D^n[(1-x)^{\alpha+n}(1+x)^{\beta+n}]$$

where

$$\lambda = \alpha+\beta+1 \qquad D \equiv \frac{\mathrm{d}}{\mathrm{d}x}.$$

The ultraspherical polynomials are Jacobi polynomials with $\alpha = \beta$ and are thus a subclass of the Jacobi polynomials. It is convenient to weigh the ultraspherical polynomials so that

$$C_n^{(\alpha)}(x) = \frac{n!\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma\left(n+\alpha+\frac{1}{2}\right)} P_n^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(x) \tag{2}$$

which gives $C_n^{(\alpha)}(1) = 1$ $(n = 0, 1, 2, \ldots)$; this is not the usual standardization, but has the desirable properties that $C_n^{(0)}(x) = T_n(x)$, $C_n^{(\frac{1}{2})}(x) = P_n(x)$, and $C_n^{(1)}(x) = (1/(n+1))U_n(x)$, where $T_n(x)$, $U_n(x)$ and $P_n(x)$ are Chebyshev polynomials of the first and second kinds and Legendre polynomials respectively.

Let $f(x)$ be an infinitely differentiable function defined on $[-1, 1]$, then we can write

$$f(x) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) \tag{3}$$

and for the qth derivative of $f(x)$,

$$f^{(q)}(x) = \sum_{n=0}^{\infty} a_n^{(q)} P_n^{(\alpha,\beta)}(x) \qquad a_n^{(0)} = a_n \tag{4}$$

it is possible to derive a recurrence relation involving the Jacobi coefficients of successive derivatives of $f(x)$. Let us write

$$\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{\infty} a_n^{(q-1)} P_n^{(\alpha,\beta)}(x) = \sum_{n=0}^{\infty} a_n^{(q)} P_n^{(\alpha,\beta)}(x)$$

then use of the identity

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) = \frac{2}{(n+\lambda-1)(2n+\lambda-1)_3}\Big[&(n+\lambda-1)_2(2n+\lambda-1)DP_{n+1}^{(\alpha,\beta)}(x) \\ &+ (\alpha-\beta)(n+\lambda-1)(2n+\lambda)DP_n^{(\alpha,\beta)}(x) \\ &- (n+\alpha)(n+\beta)(2n+\lambda+1)DP_{n-1}^{(\alpha,\beta)}(x)\Big] \qquad n \geqslant 1 \end{aligned} \tag{5}$$

leads to the recurrence relation

$$\frac{(n+\lambda-1)}{(2n+\lambda-1)(2n+\lambda-2)}a_{n-1}^{(q)}+\frac{(\alpha-\beta)}{(2n+\lambda+1)(2n+\lambda-1)}a_n^{(q)}$$
$$-\frac{(n+\alpha+1)(n+\beta+1)}{(2n+\lambda+2)(2n+\lambda+1)(n+\lambda)}a_{n+1}^{(q)}=\frac{1}{2}a_n^{(q-1)} \qquad q\geqslant 1 \quad n\geqslant 1 \tag{6}$$

where $a_n^{(0)}=a_n$ and $(\alpha)_k=\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$ is the Pochhammer symbol.

Theorem 1.

$$D^q P_n^{(\alpha,\beta)}(x)=2^{-q}(n+\lambda)_q\sum_{i=0}^{n-q}C_{n-q,i}(\alpha+q,\beta+q,\alpha,\beta)P_i^{(\alpha,\beta)}(x) \tag{7}$$

where

$$C_{n-q,i}(\alpha+q,\beta+q,\alpha,\beta)=\frac{(n+q+\lambda)_i(i+q+\alpha+1)_{n-i-q}\Gamma(i+\lambda)}{(n-i-q)!\Gamma(2i+\lambda)}$$
$$\times\,{}_3F_2\left(\begin{matrix}-n+q+i, & n+i+q+\lambda, & i+\alpha+1\\ i+q+\alpha+1, & 2i+\lambda+1 & \end{matrix};1\right).$$

Theorem 2.

$$a_n^{(q)}=2^{-q}\sum_{i=0}^{\infty}(n+i+q+\lambda-1)_q C_{n+i,n}(\alpha+q,\beta+q,\alpha,\beta)a_{n+i+q} \qquad n\geqslant 0 \quad q\geqslant 1 \tag{8}$$

where

$$C_{n+i,n}(\alpha+q,\beta+q,\alpha,\beta)=\frac{(n+i+2q+\lambda-1)_n(n+\alpha+q+1)_i\Gamma(n+\lambda)}{i!\Gamma(2n+\lambda)}$$
$$\times\,{}_3F_2\left(\begin{matrix}-i, & 2n+i+2q+\lambda, & n+\alpha+1\\ n+q+\alpha+1, & 2n+\lambda+1 & \end{matrix};1\right).$$

Let $b_n^{(q)}$, $q\geqslant 1$, denote the Jacobi expansion coefficients of $u(x)$, $x\in[-1,1]$, i.e.,

$$u(x)=\sum_{n=0}^{\infty}b_n^{(q)}P_n^{(\alpha,\beta)}(x) \tag{9}$$

and let $u(x)$ be an infinitely differentiable function, then we may express the ℓth derivative of $u(x)$ in the form

$$u^{(\ell)}(x)=\sum_{n=0}^{\infty}b_n^{(q-\ell)}P_n^{(\alpha,\beta)}(x) \qquad \ell\geqslant 0 \tag{10}$$

and in particular

$$u^{(q)}(x)=\sum_{n=0}^{\infty}b_nP_n^{(\alpha,\beta)}(x) \qquad b_n=b_n^{(0)} \tag{11}$$

from (9) and (10) we can write

$$\sum_{n=0}^{\infty}b_n^{(q)}\frac{\mathrm{d}}{\mathrm{d}x}P_n^{(\alpha,\beta)}(x)=\sum_{n=0}^{\infty}b_n^{(q-1)}P_n^{(\alpha,\beta)}(x) \tag{12}$$

then use of identity (5) with equation (12), leads to the recurrence relation

$$\frac{1}{2}b_n^{(q)} = \frac{(n+\lambda-1)}{(2n+\lambda-2)(2n+\lambda-1)}b_{n-1}^{(q-1)} + \frac{(\alpha-\beta)}{(2n+\lambda-1)(2n+\lambda+1)}b_n^{(q-1)} - \frac{(n+\alpha+1)(n+\beta+1)}{(n+\lambda)(2n+\lambda+1)(2n+\lambda+2)}b_{n+1}^{(q-1)} \qquad q \geqslant 1 \quad n \geqslant 1. \tag{13}$$

Theorem 3. *If we define the q times repeated integration of* $P_n^{(\alpha,\beta)}(x)$ *by*

$$I_n^{(q,\alpha,\beta)}(x) = \overbrace{\int\int\cdots\int}^{q\,\text{times}} P_n^{(\alpha,\beta)}(x)\,\mathrm{d}x\,\mathrm{d}x\cdots\mathrm{d}x \tag{14}$$

then

$$I_n^{(q,\alpha,\beta)}(x) = \frac{2^q}{(n-q+\lambda)_q}\sum_{k=q}^{n+q} C_{n+q,k}(\alpha-q,\beta-q,\alpha,\beta)P_k^{(\alpha,\beta)}(x) + \pi_{q-1}(x) \tag{15}$$

$q \geqslant 0, n \geqslant q+1$ *for* $\alpha = \beta = -\frac{1}{2}$ $\qquad q \geqslant 0, n \geqslant q$ *for* $\alpha \neq -\frac{1}{2}$ *or* $\beta \neq -\frac{1}{2}$

where

$$C_{n+q,k}(\alpha-q,\beta-q,\alpha,\beta) = \frac{(n-q+\lambda)_k(k-q+\alpha+1)_{n-k+q}\Gamma(k+\lambda)}{(n-k+q)!\Gamma(2k+\lambda)} \times {}_3F_2\left(\begin{matrix} -n-q+k, & n+k-q+\lambda, & k+\alpha+1 \\ k-q+\alpha+1, & 2k+\lambda+1 & \end{matrix}; 1\right)$$

and $\pi_{q-1}(x)$ *is a polynomial of degree at most* $(q-1)$.

Theorem 4. *Let* $u(x)$ *be an infinitely differentiable function defined on the interval* $[-1, 1]$. *Then the Jacobi coefficients* $b_n^{(q)}$ *of* $u(x)$ *are related to the Jacobi coefficients* b_n *of the qth derivative of* $u(x)$ *by*

$$b_n^{(q)} = 2^q \sum_{j=0}^{\infty} \frac{b_{n+j-q}}{(n+j-2q+\lambda)_q} C_{j+n,n}(\alpha-q,\beta-q,\alpha,\beta) \qquad n \geqslant q. \tag{16}$$

Remark 1. It is worth mentioning that we have demonstrated how the differentiated and integrated expansions (4) and (9) can be applied to solve two-point boundary value problems by using the spectral Galerkin method. The interested reader is referred to Doha (2002b, 2003a).

3. Jacobi coefficients of the moments of one single Jacobi polynomial of any degree

For the evaluation of Jacobi coefficients of the moments of high-order derivatives of infinitely differentiable functions, the following theorem is needed.

Theorem 5.

$$x^m P_j^{(\alpha,\beta)}(x) = \sum_{n=0}^{2m} a_{mn}(j) P_{j+m-n}^{(\alpha,\beta)}(x) \qquad m \geqslant 0 \quad j \geqslant 0 \tag{17}$$

with $P_{-r}^{(\alpha,\beta)}(x) = 0,\, r \geqslant 1$, *where*

$$a_{mn}(j) = \frac{(-1)^n 2^{j+m-n} m!(2j+2m-2n+\lambda)\Gamma(j+m-n+\lambda)\Gamma(j+\alpha+1)\Gamma(j+\beta+1)}{\Gamma(j+m-n+\alpha+1)\Gamma(j+m-n+\beta+1)\Gamma(j+\lambda)}$$
$$\times \sum_{k=\max(0,j-n)}^{\min(j+m-n,j)} \binom{j+m-n}{k} \frac{\Gamma(j+k+\lambda)}{2^k(n+k-j)!\Gamma(3j+2m-2n-k+\lambda+1)}$$
$$\times \sum_{\ell=0}^{j-k} \frac{(-1)^\ell \Gamma(2j+m-n-k-\ell+\alpha+1)\Gamma(j+m+\ell-n+\beta+1)}{\ell!(j-k-\ell)!\Gamma(j-\ell+\alpha+1)\Gamma(k+\ell+\beta+1)}$$
$$\times {}_2F_1(j-k-n,\, j+m+\ell-n+\beta+1;\, 3j+2m-2n-k+\lambda+1;\, 2). \quad (18)$$

Proof. We use the induction principle to prove this theorem. In view of recurrence relation

$$x P_j^{(\alpha,\beta)}(x) = \frac{2(j+1)(j+\lambda)}{(2j+\lambda)(2j+\lambda+1)} P_{j+1}^{(\alpha,\beta)}(x) - \frac{(\alpha^2-\beta^2)}{(2j+\lambda-1)(2j+\lambda+1)} P_j^{(\alpha,\beta)}(x)$$
$$+ \frac{2(j+\alpha)(j+\beta)}{(2j+\lambda-1)(2j+\lambda)} P_{j-1}^{(\alpha,\beta)}(x) \qquad j \geqslant 0$$

we may write

$$x P_j^{(\alpha,\beta)(x)} = a_{10}(j) P_{j+1}^{(\alpha,\beta)}(x) + a_{11}(j) P_j^{(\alpha,\beta)}(x) + a_{12}(j) P_{j-1}^{(\alpha,\beta)}(x) \quad (19)$$

and this in turn shows that (17) is true for $m = 1$. Proceeding by induction, assuming that (17) is valid for m, we want to prove that

$$x^{m+1} P_j^{(\alpha,\beta)}(x) = \sum_{n=0}^{2m+2} a_{m+1,n}(j) P_{j+m-n+1}^{(\alpha,\beta)}(x). \quad (20)$$

From (19) and assuming the validity of (17) for m, we have

$$x^{m+1} P_j^{(\alpha,\beta)}(x) = \sum_{n=0}^{2m} a_{mn}(j)\big[a_{10}(j+m-n) P_{j+m-n+1}^{(\alpha,\beta)}(x) + a_{11}(j+m-n) P_{j+m-n}^{(\alpha,\beta)}(x)$$
$$+ a_{12}(j+m-n) P_{j+m-n-1}^{(\alpha,\beta)}(x)\big].$$

Collecting similar terms, we get

$$x^{m+1} P_j^{(\alpha,\beta)}(x) = a_{m0}(j) a_{10}(j+m) P_{j+m+1}^{(\alpha,\beta)}(x) + [a_{m1}(j) a_{10}(j+m-1)$$
$$+ a_{m0}(j) a_{11}(j+m)] P_{j+m}^{(\alpha,\beta)}(x) + \sum_{n=2}^{2m} [a_{mn}(j) a_{10}(j+m-n)$$
$$+ a_{m,n-1}(j) a_{11}(j+m-n+1) + a_{m,n-2}(j) a_{12}(j+m-n+2)] P_{j+m-n+1}^{(\alpha,\beta)}(x)$$
$$+ [a_{m,2m}(j) a_{11}(j-m) + a_{m,2m-1}(j) a_{12}(j-m+1)] P_{j-m}^{(\alpha,\beta)}(x)$$
$$+ a_{m,2m}(j) a_{12}(j-m) P_{j-m-1}^{(\alpha,\beta)}(x). \quad (21)$$

It can be easily shown that

$$a_{m+1,0}(j) = a_{m0}(j) a_{10}(j+m)$$
$$a_{m+1,1}(j) = a_{m1}(j) a_{10}(j+m-1) + a_{m0}(j) a_{11}(j+m)$$
$$a_{m+1,n}(j) = a_{mn}(j) a_{10}(j+m-n) + a_{m,n-1}(j) a_{11}(j+m-n+1)$$
$$+ a_{m,n-2}(j) a_{12}(j+m-n+2)$$
$$a_{m+1,2m+1}(j) = a_{m,2m}(j) a_{11}(j-m) + a_{m,2m-1}(j) a_{12}(j-m+1)$$
$$a_{m+1,2m+2}(j) = a_{m,2m}(j) a_{12}(j-m)$$

and accordingly, formula (21) becomes

$$x^{m+1} P_j^{(\alpha,\beta)}(x) = \sum_{n=0}^{2m+2} a_{m+1,n}(j) P_{j+m-n+1}^{(\alpha,\beta)}(x)$$

which completes the induction and proves the theorem. □

Corollary 1. *It can be easily shown that the expansion coefficients $a_{mn}(j)$ of theorem 5 satisfy the recurrence relation*

$$a_{mn}(j) = \sum_{k=0}^{2} a_{m-1,n+k-2}(j) a_{1,2-k}(j+m-n-k+1) \qquad n = 0, 1, \ldots, 2m \tag{22}$$

where

$$a_{1k}(j) = \begin{cases} \dfrac{2(j+1)(j+\lambda)}{(2j+\lambda)(2j+\lambda+1)} & k = 0 \\ \dfrac{-(\alpha^2-\beta^2)}{(2j+\lambda-1)(2j+\lambda+1)} & k = 1 \quad a_{00}(j) = 1 \\ \dfrac{2(j+\alpha)(j+\beta)}{(2j+\lambda-1)(2j+\lambda)} & k = 2 \end{cases} \tag{23}$$

with

$$a_{m-1,-\ell}(j) = 0 \qquad \forall \ell \geqslant 0 \qquad a_{m-1,r}(j) = 0 \qquad r = 2m-1, 2m.$$

Corollary 2. *It is not difficult to show that*

$$x^m P_j^{(\alpha,\beta)}(x) = \sum_{n=0}^{j+m} a_{m,j+m-n}(j) P_n^{(\alpha,\beta)}(x) \qquad j \geqslant 0 \quad m \geqslant 0 \tag{24}$$

and

$$x^m = \sum_{n=0}^{m} a_{m,m-n}(0) P_n^{(\alpha,\beta)}(x) \qquad m \geqslant 0 \tag{25}$$

where

$$a_{m,m-n}(0) = \frac{(-1)^{m-n} 2^n m! \Gamma(n+\lambda)}{(m-n)! \Gamma(2n+\lambda)} \, {}_2F_1(-(m-n), n+\beta+1; 2n+\lambda+1; 2) \tag{26}$$

which is in agreement with Luke (1975), p 440, formula (1).

Corollary 3. *The expansion of $x^m C_j^{(\alpha)}(x)$ in series of ultraspherical polynomials is given by*

$$x^m C_j^{(\alpha)}(x) = \sum_{n=0}^{m+j} b_{m,j+m-n}(j) C_n^{(\alpha)}(x) \tag{27}$$

where

$$\begin{aligned} b_{m,j+m-n}(j) = {} & \frac{(-1)^{j+m-n} 2^{n+1} j! m! (n+\alpha) \Gamma(n+2\alpha) \Gamma\left(j+\alpha+\frac{1}{2}\right)}{n! \Gamma(j+2\alpha) \Gamma\left(n+\alpha+\frac{1}{2}\right)} \\ & \times \sum_{k=\max(0,n-m)}^{\min(n,j)} \binom{n}{k} \frac{\Gamma(j+k+2\alpha)}{2^k (k+m-n)! \Gamma(j+2n-k+2\alpha+1)} \\ & \times \sum_{\ell=0}^{j-k} \frac{(-1)^\ell \Gamma\left(j+n-k-\ell+\alpha+\frac{1}{2}\right) \Gamma\left(\ell+n+\alpha+\frac{1}{2}\right)}{\ell! (j-k-\ell)! \Gamma\left(j-\ell+\alpha+\frac{1}{2}\right) \Gamma\left(k+\ell+\alpha+\frac{1}{2}\right)} \\ & \times {}_2F_1\left(-(k+m-n), \ell+n+\alpha+\tfrac{1}{2}; j+2n-k+2\alpha+1; 2\right). \end{aligned}$$

Proof. Expansion (27) follows directly from equation (24), as a special case, by taking $\alpha = \beta$ and each is replaced by $\left(\alpha - \frac{1}{2}\right)$, then using relation (2). □

Remark 2. The expansions of $x^m\ T_j(x)$, $x^m U_j(x)$ and $x^m P_j(x)$ in series of Chebyshev polynomials of the first and second kinds and of Legendre polynomials follow from equation (27) by taking $\alpha = 0$, 1 and $\frac{1}{2}$ respectively.

Corollary 4. *The expansion of $x^{2r+\epsilon}$ in series of ultraspherical polynomials is given by*

$$x^{2r+\epsilon} = \frac{(2r+\epsilon)!\Gamma\left(\frac{1}{2}\right)}{2^{2r+2\alpha+\epsilon-1}\Gamma\left(\alpha+\frac{1}{2}\right)} \sum_{s=0}^{r} \frac{(2s+\alpha+\epsilon)\Gamma(2s+2\alpha+\epsilon)}{(r-s)!(2s+\epsilon)!\Gamma(r+s+\alpha+\epsilon+1)} C_{2s+\epsilon}^{(\alpha)}(x)$$
$$\epsilon = 0, 1.$$

Proof. Set $j = 0$ in equation (27), and after performing some manipulation, we get

$$x^m = \sum_{n=0}^{m} b_{m,m-n}(0) C_n^{(\alpha)}(x) \tag{28}$$

where

$$b_{m,m-n}(0) = \frac{(-1)^{m-n} m!\Gamma\left(\frac{1}{2}\right)\Gamma(n+2\alpha)}{n!(m-n)!2^{n+2\alpha-1}\Gamma(n+\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)} \times {}_2F_1\left(-(m-n), n+\alpha+\frac{1}{2}; 2n+2\alpha+1; 2\right).$$

Taking into account that (Luke (1975), p 272, formulae (12) and (13))

$${}_2F_1(-N, a; 2a; 2) = \begin{cases} 0 & \text{for} \quad N \text{ odd} \\ \dfrac{\left(\frac{1}{2}\right)_{\frac{N}{2}}}{\left(a+\frac{1}{2}\right)_{\frac{N}{2}}} & \text{for} \quad N \text{ even} \end{cases} \tag{29}$$

for $a \neq 0, -1, -2, \ldots$, we see that $b_{m,m-n}(0) = 0$ when $(m-n)$ is odd, while

$$b_{m,m-n}(0) = \frac{m!(n+\alpha)\Gamma(n+2\alpha)\Gamma\left(\frac{m-n+1}{2}\right)}{n!(m-n)!2^{n+2\alpha-1}\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{m+n}{2}+\alpha+1\right)} \qquad (m-n) \text{ even}$$

which using the Legendre duplication formula

$$\Gamma(2z) = \frac{2^{2z-1}\Gamma\left(z+\frac{1}{2}\right)\Gamma(z)}{\Gamma\left(\frac{1}{2}\right)}$$

can be written in the form

$$b_{m,m-n}(0) = \frac{m!(n+\alpha)\Gamma\left(\frac{1}{2}\right)\Gamma(n+2\alpha)}{n!\left(\frac{m-n}{2}\right)!2^{m+2\alpha-1}\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{m+n}{2}+\alpha+1\right)} \qquad (m-n) \text{ even.}$$

Finally, writing $m = 2r + \epsilon$, $n = 2s + \epsilon$, where r, s are integers and $\epsilon = 0, 1$ for m even and odd respectively, expansion (28) reads in this case

$$x^{2r+\epsilon} = \frac{(2r+\epsilon)!\Gamma\left(\frac{1}{2}\right)}{2^{2r+2\alpha+\epsilon-1}\Gamma\left(\alpha+\frac{1}{2}\right)} \sum_{s=0}^{r} \frac{(2s+\alpha+\epsilon)\Gamma(2s+2\alpha+\epsilon)}{(r-s)!(2s+\epsilon)!\Gamma(r+s+\alpha+\epsilon+1)} C_{2s+\epsilon}^{(\alpha)}(x)$$

and this completes the proof of the corollary. □

Remark 3. The expansion of x^m in series of Chebyshev polynomials of the first kind $T_n(x)$ and of the second kind $U_n(x)$ and Legendre polynomials $P_n(x)$ follows from equation (28) directly by taking $\alpha = 0$, 1 and $\frac{1}{2}$ respectively.

4. Jacobi coefficients of the moments of a general-order derivative of an infinitely differentiable function

Under the assumptions of (4) and (17), and for a positive integer ℓ, let

$$x^{\ell}\frac{\mathrm{d}^q f(x)}{\mathrm{d}x^q} = I^{q,\ell} \tag{30}$$

and if we write

$$I^{q,\ell} = \sum_{i=0}^{\infty} b_i^{q,\ell} P_i^{(\alpha,\beta)}(x) \tag{31}$$

then

Theorem 6

$$b_i^{q,\ell} = \begin{cases} \sum_{k=0}^{\ell-1} a_{\ell,k+\ell-i}(k)a_k^{(q)} + \sum_{k=0}^{i} a_{\ell,k+2\ell-i}(k+\ell)a_{k+\ell}^{(q)} & 0 \leqslant i \leqslant \ell \\ \sum_{k=i-\ell}^{\ell-1} a_{\ell,k+\ell-i}(k)a_k^{(q)} + \sum_{k=0}^{i} a_{\ell,k+2\ell-i}(k+\ell)a_{k+\ell}^{(q)} & \ell+1 \leqslant i \leqslant 2\ell-1 \\ \sum_{k=i-2\ell}^{i} a_{\ell,k+2\ell-i}(k+\ell)a_{k+\ell}^{(q)} & i \geqslant 2\ell. \end{cases} \tag{32}$$

Proof. Equations (7) and (17) and (30) give

$$I^{q,\ell} = \sum_{k=0}^{\infty} a_k^{(q)} \sum_{j=0}^{2\ell} a_{\ell,j}(k) P_{k+\ell-j}^{(\alpha,\beta)}(x). \tag{33}$$

By letting $i = k+\ell-j$, then (33) may be written in the form

$$\begin{aligned} I^{q,\ell} &= \sum_{k=0}^{\ell-1} a_k^{(q)} \sum_{i=k-\ell}^{k+\ell} a_{\ell,k+\ell-i}(k) P_i^{(\alpha,\beta)}(x) + \sum_{k=\ell}^{\infty} a_k^{(q)} \sum_{i=k-\ell}^{k+\ell} a_{\ell,k+\ell-i}(k) P_i^{(\alpha,\beta)}(x) \\ &= \sum_1 + \sum_2 \end{aligned} \tag{34}$$

where

$$\sum_1 = \sum_{k=0}^{\ell-1} a_k^{(q)} \sum_{i=k-\ell}^{k+\ell} a_{\ell,k+\ell-i}(k) P_i^{(\alpha,\beta)}(x)$$

$$\sum_2 = \sum_{k=\ell}^{\infty} a_k^{(q)} \sum_{i=k-\ell}^{k+\ell} a_{\ell,k+\ell-i}(k) P_i^{(\alpha,\beta)}(x).$$

Considering $\sum_1$ first,

$$\begin{aligned} \sum_1 &= \sum_{k=0}^{\ell-1} a_k^{(q)} \sum_{i=k-\ell}^{-1} a_{\ell,k+\ell-i}(k) P_i^{(\alpha,\beta)}(x) + \sum_{k=0}^{\ell-1} a_k^{(q)} \sum_{i=0}^{k+\ell} a_{\ell,k+\ell-i}(k) P_i^{(\alpha,\beta)}(x) \\ &= \sum_{11} + \sum_{12}. \end{aligned} \tag{35}$$

Clearly,

$$\sum_{11} = \sum_{k=0}^{\ell-1} a_k^{(q)} \sum_{i=k-\ell}^{-1} a_{\ell,k+\ell-i}(k) P_i^{(\alpha,\beta)}(x) = \sum_{k=0}^{\ell-1} a_k^{(q)} \sum_{i=1}^{\ell-k} a_{\ell,k+\ell+i}(k) P_{-i}^{(\alpha,\beta)}(x)$$

hence

$$\sum_{11} = 0. \tag{36}$$

Now,

$$\begin{aligned}\sum_{12} &= \sum_{k=0}^{\ell-1} a_k^{(q)} \sum_{i=0}^{k+\ell} a_{\ell,k+\ell-i}(k) P_i^{(\alpha,\beta)}(x) \\ &= \sum_{i=0}^{\ell} \sum_{k=0}^{\ell-1} a_k^{(q)} a_{\ell,k+\ell-i}(k) P_i^{(\alpha,\beta)}(x) + \sum_{i=\ell+1}^{2\ell-1} \sum_{k=i-\ell}^{\ell-1} a_k^{(q)} a_{\ell,k+\ell-i}(k) P_i^{(\alpha,\beta)}(x)\end{aligned}$$

hence

$$\sum_{12} = \sum_{i=0}^{2\ell-1} \sum_{k=\max(0,i-\ell)}^{\ell-1} a_k^{(q)} a_{\ell,k+\ell-i}(k) P_i^{(\alpha,\beta)}(x). \tag{37}$$

Substitution of (36) and (37) into (35) yields

$$\sum_{1} = \sum_{i=0}^{2\ell-1} \sum_{k=\max(0,i-\ell)}^{\ell-1} a_k^{(q)} a_{\ell,k+\ell-i}(k) P_i^{(\alpha,\beta)}(x). \tag{38}$$

If when considering $\sum_2$, one takes $k+\ell$ instead of k, then it is not difficult to show that

$$\sum_{2} = \sum_{i=0}^{\infty} \sum_{k=\max(0,i-2\ell)}^{i} a_{k+\ell}^{(q)} a_{\ell,k+2\ell-i}(k+\ell) P_i^{(\alpha,\beta)}(x). \tag{39}$$

Substitution of (38) and (39) into (34) gives the required results of (32) and completes the proof of theorem 6. □

5. Recurrence relations for connection coefficients between different Jacobi polynomials

In this section we consider the problem of determining the connection coefficients between different polynomial systems. Suppose V is a vector space of all polynomials over the real or complex numbers and V_m is the subspace of polynomials of degree less than or equal to m. Suppose $p_0(x), p_1(x), p_2(x), \ldots$ is a sequence of polynomials such that $p_n(x)$ is of exact degree n; let $q_0(x), q_1(x), q_2(x), \ldots$ be another such sequence. Clearly, these sequences form a basis for V. It is also evident that $p_0(x), p_1(x), \ldots, p_m(x)$ and $q_0(x), q_1(x), \ldots, q_m(x)$ give two bases for V_m. While working with finite-dimensional vector spaces, it is often necessary to find the matrix that transforms a basis of a given space to another basis. This means that one is interested in the connection coefficients $a_i(n)$ that satisfy

$$q_n(x) = \sum_{i=0}^{n} a_i(n) p_i(x). \tag{40}$$

The choice of $p_n(x)$ and $q_n(x)$ depends on the situation. For example, suppose

$$p_n(x) = x^n \qquad q_n(x) = x(x-1)\cdots(x-n+1) = (-1)^n(-x)_n = \frac{\Gamma(x+1)}{\Gamma(x-n+1)}$$

then the connection coefficients $a_i(n)$ are Stirling numbers of the first kind. If the roles of these $p_n(x)$ and $q_n(x)$ are interchanged, then we get Stirling numbers of the second kind. These numbers are useful in some combinatorial polynomials (see Abramowitz and Stegun (1970), pp 824–5).

The connection coefficients between many of the classical orthogonal polynomial systems have been determined by different kinds of methods (see, e.g., Szegö (1985), Rainville (1960), Andrews *et al* (1999)). The aim of this section is to describe a simple procedure (based on the results of theorem 6) in order to find recurrence relations, sometimes easy to solve, between the coefficients $a_i(n)$ when $p_i(x) = P_i^{(\gamma,\delta)}(x)$ and $q_i(x) = P_i^{(\alpha,\beta)}(x)$, where $P_i^{(.,.)}(x)$ are the Jacobi orthogonal polynomials. This gives an alternative and different way to compare the approaches of Askey and Gasper (1971), Ronveaux *et al* (1995, 1996), Area *et al* (1998), Godoy *et al* (1997), Koepf and Schmersau (1998), Lewanowicz (2002), Lewanowicz and Woźny (2001), Lewanowicz *et al* (2000), Sánchez-Ruiz and Dehesa (1998). A nonrecursive way to approach the problem in the case of classical orthogonal polynomials of discrete variable can be found in Gasper (1974). Moreover, other authors have considered the problem from a recursive point of view (see Koepf and Schmersau (1988)), or even in classical discrete and q-analogues (cf Álvarez-Nodarse *et al* (1998), Álvarez-Nodarse and Ronveaux (1996)). Since the connection coefficients $a_i(n)$ depend on two parameters i and n, the most interesting recurrence relations are those which leave one of the parameters fixed. The success of our procedure depends heavily on whether or not these recurrence relations are of minimal order, i.e. the shortest ones in order. In cases when the order of the resulting recurrence relation is 1, it defines a hypergeometric term which can be given explicitly in terms of the Pochhammer symbol $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$.

5.1. The Jacobi–Jacobi connection problem

The link between $P_n^{(\gamma,\delta)}(x)$ and $P_i^{(\alpha,\beta)}(x)$ given by (40) can easily be replaced by a linear relation involving only $P_i^{(\alpha,\beta)}(x)$ using the Jacobi differential equation, namely

$$(1-x^2)D^2P_n^{(\gamma,\delta)}(x) + [\delta - \gamma - (2+\delta+\gamma)x]DP_n^{(\gamma,\delta)}(x) + n(1+\gamma+\delta+n)P_n^{(\gamma,\delta)}(x) = 0 \tag{41}$$

by substituting

$$P_n^{(\gamma,\delta)}(x) = \sum_{i=0}^{\infty} a_i(n) P_i^{(\alpha,\beta)}(x) \tag{42}$$

with $a_{n+1}(n) = a_{n+2}(n) = \cdots = 0$. By virtue of formulae (30) and (31), equation (42) takes the form

$$I^{2,0} - I^{2,2} - (2+\gamma+\delta)I^{1,1} + (\delta-\gamma)I^{1,0} + n(1+\gamma+\delta+n)I^{0,0} = 0$$

or

$$b_i^{2,0} - b_i^{2,2} - (2+\gamma+\delta)b_i^{1,1} + (\delta-\gamma)b_i^{1,0} + n(1+\gamma+\delta+n)b_i^{0,0} = 0. \tag{43}$$

Formula (32) gives

$$b_i^{2,0} = a_i^{(2)}(n) \qquad b_i^{1,0} = a_i^{(1)}(n) \qquad b_i^{0,0} = a_i(n) \qquad i \geqslant 0 \tag{44}$$

$$b_i^{1,1} = \sum_{k=i-2}^{i} a_{1,k+2-i}(k+1)a_{k+1}^{(1)}(n) \qquad b_i^{2,2} = \sum_{k=i-4}^{i} a_{2,k+4-i}(k+2)a_{k+2}^{(2)}(n) \qquad i \geqslant 0. \tag{45}$$

Substitution of relations (44) and (45) into (43), and making use of formulae (8) and (18)—after some manipulation—yields the following recurrence relation:

$$\frac{2(n-i)(i+\lambda)(i+\lambda+1)(n+i+\gamma+\delta+1)}{(2i+\lambda)(2i+\lambda+1)}a_i(n)$$
$$= (i+\lambda+1)\left\{\gamma-\delta+\beta-\alpha+\frac{(\beta-\alpha)(\lambda+1)(\gamma+\delta-\alpha-\beta)}{(2i+\lambda+1)(2i+\lambda+3)}\right.$$
$$\left.+\frac{2(\beta-\alpha)[n(n+\gamma+\delta+1)-(i+1)(i+\lambda+1)]}{(2i+\lambda+1)(2i+\lambda+3)}\right\}a_{i+1}(n)$$
$$+\frac{2(i+\alpha+2)(i+\beta+2)(n-i+\gamma+\delta-\lambda-1)(n+i+\lambda+2)}{(2i+\lambda+3)(2i+\lambda+4)}a_{i+2}(n)$$
$$i = n-1, n-2, \ldots, 0 \tag{46}$$

which is of order 2. It is to be noted here that the second-order recurrence relation (46) generates the coefficients $a_i(n) = a_i(n; \alpha, \beta, \gamma, \delta)$ of (42) by recurring backwards with the initial conditions given by

$$a_{n+1}(n) = a_{n+2}(n) = 0 \qquad \text{and} \qquad a_n(n) = \frac{\Gamma(n+\lambda)\Gamma(2n+\gamma+\delta+1)}{\Gamma(2n+\lambda)\Gamma(n+\gamma+\delta+1)}.$$

The coefficient $a_n(n)$, which only depends on the relative normalization of $P_n^{(\gamma,\delta)}(x)$ and $P_n^{(\alpha,\beta)}(x)$, has been easily obtained by identification of the highest power in expansion (42). The solution of (46) is

$$a_i(n;\alpha,\beta,\gamma,\delta) = \frac{\Gamma(n+\gamma+1)\Gamma(i+\lambda)\Gamma(n+i+\gamma+\delta+1)}{\Gamma(i+\gamma+1)\Gamma(n+\gamma+\delta+1)\Gamma(2i+\lambda)(n-i)!}$$
$$\times {}_3F_2(-(n-i), i+\alpha+1, n+i+\gamma+\delta+1; i+\gamma+1, 2i+\lambda+1; 1) \tag{47}$$

which is in complete agreement with that given in Andrews *et al* (1999), p 357.

If we take $\gamma = \alpha$, then the ${}_3F_2$ reduces to a terminating ${}_2F_1$, which can be evaluated by the Chu–Vandemonde formula

$${}_2F_1(-n, a, c; 1) = \frac{(c-a)_n}{(c)_n}.$$

The ${}_3F_2$ can be again summed if $\delta = \beta$. Since in this case we get a balanced ${}_3F_2$ whose value is given by a Pfaff–Saalschütz identity

$${}_3F_2(-n, a, b; c, 1+a+b-c-n; 1) = \frac{(c-a)_n(c-b)_n}{n!(c)_n}.$$

Corollary 5. *In this connection problem* ($\gamma = \alpha$)

$$P_n^{(\alpha,\delta)}(x) = \sum_{i=0}^{n} a_i(n) P_i^{(\alpha,\beta)}(x) \tag{48}$$

the coefficients $a_i(n)$ *are given by*

$$a_i(n) = \frac{(-1)^{n-i}(\alpha+1)_n(\delta-\beta)_{n-i}(\lambda)_i(2i+\lambda)(\alpha+\delta+n+1)_i}{(n-i)!(\alpha+1)_i(\lambda)_{n+1}(n+\lambda+1)_i}. \tag{49}$$

Corollary 6. *In this connection problem* ($\delta = \beta$)

$$P_n^{(\gamma,\beta)}(x) = \sum_{i=0}^{n} a_i(n) P_i^{(\alpha,\beta)}(x) \tag{50}$$

the coefficients $a_i(n)$ are given by

$$a_i(n) = \frac{(2i+\lambda)(\beta+1)_n(\lambda)_i(n+\beta+\gamma+1)_i(\gamma-\alpha)_{n-i}}{(n-i)!(\beta+1)_i(\lambda)_{n+1}(n+\lambda+1)_i}. \tag{51}$$

Corollary 7. *In this connection problem* ($\delta = \gamma, \beta = \alpha$)

$$P_n^{(\gamma,\gamma)}(x) = \sum_{i=0}^{n} a_i(n) P_i^{(\alpha,\alpha)}(x) \tag{52}$$

the coefficients $a_i(n)$ are given by

$$a_i(n) = \frac{(\gamma+1)_n(2\alpha+1)_i\left(\gamma+\frac{1}{2}\right)_{\frac{n+i}{2}}\left(\alpha+\frac{3}{2}\right)_i(\gamma-\alpha)_{\frac{n-i}{2}}}{(2\gamma+1)_n(\alpha+1)_i\left(\alpha+\frac{1}{2}\right)_i\left(\alpha+\frac{3}{2}\right)_{\frac{n+i}{2}}\left(\frac{n-i}{2}\right)!} \qquad (n-i) \text{ even}. \tag{53}$$

5.2. The ultraspherical–ultraspherical connection problem

The link between $C_n^{(\nu)}(x)$ and $C_i^{(\mu)}(x)$ can be easily obtained by taking $\gamma = \delta = \nu - \frac{1}{2}, \alpha = \beta = \mu - \frac{1}{2}$ in (42) and returning to relation (2), to get the connection problem

$$C_n^{(\nu)}(x) = \sum_{i=0}^{n} b_i(n) C_i^{(\mu)}(x) \tag{54}$$

where

$$b_i(n) = \frac{n!\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(i+\mu+\frac{1}{2}\right)}{i!\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(n+\nu+\frac{1}{2}\right)} a_i(n)$$

and $a_i(n)$ satisfy the second-order recurrence relation

$$\frac{(n-i)(i+2\mu)(i+2\mu+1)(n+i+2\nu)}{(i+\mu)(2i+2\mu+1)} a_i(n) + \frac{\left(i+\mu+\frac{3}{2}\right)^2(n-i+2\nu-2\mu-2)(n+i+2\mu+2)}{(i+\mu+2)(2i+2\mu+3)} a_{i+2}(n) = 0$$

with the initial conditions

$$a_{n+1}(n) = a_{n+2}(n) = 0 \qquad \text{and} \qquad a_n(n) = \frac{\Gamma(n+2\mu)\Gamma(2n+2\nu)}{\Gamma(n+2\nu)\Gamma(2n+2\mu)}.$$

The $b_i(n)$ coefficients are given by

$$b_i(n) = \frac{n!(\nu)_{\frac{n+i}{2}}(\nu-\mu)_{\frac{n-i}{2}}(i+\mu)(2\mu)_i}{i!(2\nu)_n(\mu)_{\frac{n+i+2}{2}}\left(\frac{n-i}{2}\right)!} \qquad (n-i) \text{ even}. \tag{55}$$

It is worth noting that all the connection problems between Chebyshev polynomials of the first and second kinds and Legendre polynomials can be easily deduced by taking suitable values of the two parameters μ and ν in relations (54) and (55).

5.3. The Laguerre–Jacobi connection problem

In this problem

$$L_n^{(\gamma)}(x) = \sum_{i=0}^{n} a_i(n) P_i^{(\alpha,\beta)}(x) \tag{56}$$

when $L_n^{(\gamma)}(x)$ are Laguerre polynomials, which satisfy the differential equation

$$xD^2L_n^{(\gamma)}(x) + (1+\gamma-x)DL_n^{(\gamma)}(x) + nL_n^{(\gamma)}(x) = 0.$$

The coefficients $a_i(n)$ satisfy the fourth-order recurrence relation

$$\alpha_{in}a_i(n) + \beta_{in}a_{i+1}(n) + \gamma_{in}a_{i+2}(n) + \delta_{in}a_{i+3}(n) + \eta_{in}a_{i+4}(n) = 0 \qquad i = n-1, n-2, \ldots, 0 \tag{57}$$

where

$$\alpha_{in} = (n-i)\prod_{j=0}^{3}\left(\frac{i+\lambda+j}{2i+\lambda+j}\right)$$

$$\beta_{in} = (i+\lambda+1)(i+\lambda+2)\left[\frac{i(i+\lambda-1)}{2(2i+\lambda-2)(2i+\lambda-1)} + \frac{(1+\gamma)(i+\lambda+3)}{2(2i+\lambda+2)(2i+\lambda+3)}\right.$$
$$\left. + \frac{(\alpha-\beta)(4n-2i+\lambda+1)(i+\lambda+3)}{2(2i+\lambda+1)(2i+\lambda+2)(2i+\lambda+3)(2i+\lambda+5)}\right]$$

$$\gamma_{in} = (i+\lambda+2)(i+\lambda+3)\left[\frac{(\beta^2-\alpha^2)}{4(2i+\lambda-1)(2i+\lambda+1)} + \frac{(\alpha-\beta)}{(2i+\lambda+3)(2i+\lambda+5)}\right.$$
$$\times\left\{\frac{1+\gamma}{2} + \frac{(\alpha-\beta)(2n+\lambda+1)}{2(2i+\lambda+3)(2i+\lambda+5)}\right\} - \frac{(n+i+\lambda+2)(i+\alpha+2)(i+\beta+2)}{(2i+\lambda+2)(2i+\lambda+3)^2(2i+\lambda+4)}$$
$$\left. - \frac{(n-i-2)(i+\alpha+3)(i+\beta+3)}{(2i+\lambda+4)(2i+\lambda+5)^2(2i+\lambda+6)}\right]$$

$$\delta_{in} = (i+\lambda+3)\left[\frac{(i+\alpha+1)(i+\beta+1)(i+\beta+4)}{2(2i+\lambda+1)(2i+\lambda+2)} - \frac{(1+\gamma)(i+\alpha+3)(i+\beta+3)}{2(2i+\lambda+5)(2i+\lambda+6)}\right.$$
$$\left. - \frac{(\alpha-\beta)(i+\alpha+3)(i+\beta+3)(4n+2j+3\lambda+9)}{2(2i+\lambda+3)(2i+\lambda+5)(2i+\lambda+6)(2i+\lambda+7)}\right]$$

$$\eta_{in} = \frac{(i+\alpha+3)(i+\beta+3)(i+\alpha+4)(i+\beta+4)(n+i+\lambda+4)}{(2i+\lambda+5)(2i+\lambda+6)(2i+\lambda+7)(2i+\lambda+8)}$$

with $a_{n+s}(n) = 0$, $s = 1, 2, 3, 4$ and $a_n(n) = \frac{(-1)^n 2^n \Gamma(n+\lambda)}{\Gamma(2n+\lambda)}$. The solution of (57) is

$$a_i(n,\alpha,\beta,\gamma) = \frac{(-1)^i 2^i \Gamma(i+\lambda)(1+\gamma)_n}{\Gamma(2i+\lambda)} \sum_{k=i}^{n} \frac{1}{(n-k)!(k-i)!(1+\gamma)_k}$$
$$\times {}_2F_1(-(k-i), i+\beta+1; 2i+\lambda+1; 2). \tag{58}$$

Corollary 8. *The link between Laguerre-ultraspherical connection problem is given by*

$$L_n^{(\gamma)}(x) = \sum_{i=0}^{n} b_i(n) C_i^{(\alpha)}(x) \tag{59}$$

where

$$b_i(n) = \frac{(-1)^i (2\alpha)_i (1+\gamma)_n}{2^i i!(\alpha)_i} \sum_{\substack{k=i \\ (k-i)\text{even}}}^{n} \frac{\left(\frac{1}{2}\right)_{\frac{k-i}{2}}}{(n-k)!(k-i)!(1+\gamma)_k (i+\alpha+1)_{\frac{k-i}{2}}}. \tag{60}$$

Proof. If we take $\alpha = \beta$ and each is replaced by $\left(\alpha - \frac{1}{2}\right)$, then the ${}_2F_1$ in (58) can be evaluated by formula (29). In this particular case, relations (2) and (56) give immediately relation (59), where

$$b_i(n) = \frac{\Gamma\left(i+\alpha+\frac{1}{2}\right)}{i!\,\Gamma\left(\alpha+\frac{1}{2}\right)} a_i\left(n, \alpha-\frac{1}{2}, \alpha-\frac{1}{2}, \gamma\right)$$

and this with (58) and (29) enables one to get the required formula for $b_i(n)$ given by (60). □

5.4. The Hermite–Jacobi connection problem

In this problem

$$H_n(x) = \sum_{i=0}^{n} a_i(n) P_i^{(\alpha,\beta)}(x) \tag{61}$$

where $H_n(x)$ are Hermite polynomials, which satisfy the differential equation

$$D^2 H_n(x) - 2x\,D H_n(x) + 2n H_n(x) = 0.$$

The coefficients $a_i(n)$ satisfy the fourth-order recurrence relation

$$\alpha_{in} a_i(n) + \beta_{in} a_{i+1}(n) + \gamma_{in} a_{i+2}(n) + \delta_{in} a_{i+3}(n) + \eta_{in} a_{i+4}(n) = 0 \qquad i = n-1, n-2, \ldots, 0 \tag{62}$$

where

$$\alpha_{in} = (n-i)\prod_{j=0}^{3}\left(\frac{i+\lambda+j}{2i+\lambda+j}\right)$$

$$\beta_{in} = \frac{(\alpha-\beta)(2i-4n-\lambda-1)}{(2i+\lambda+5)}\prod_{j=1}^{2}\left(\frac{i+\lambda+j}{2i+\lambda+j}\right)$$

$$\gamma_{in} = (i+\lambda+2)(i+\lambda+3) \times\left[\frac{1}{4} - \frac{(2n+\lambda+2)(2i^2+2i(\lambda+4)+\alpha(5+4\beta)+\beta(5-\beta)-\alpha^2-12)}{(2i+\lambda+2)(2i+\lambda+3)(2i+\lambda+5)(2i+\lambda+6)}\right]$$

$$\delta_{in} = \frac{-(\alpha-\beta)(i+\alpha+3)(i+\beta+3)(i+\lambda+3)(4n+2i+3\lambda+9)}{(2i+\lambda+3)(2i+\lambda+5)(2i+\lambda+6)(2i+\lambda+7)}$$

$$\eta_{in} = \frac{2(i+\alpha+3)(i+\beta+3)(i+\alpha+4)(i+\beta+4)(n+i+\lambda+4)}{(2i+\lambda+5)(2i+\lambda+6)(2i+\lambda+7)(2i+\lambda+8)}$$

with $a_{n+s}(n) = 0$, $s = 1, 2, 3, 4$ and $a_n(n) = \frac{2^{2n} n!(\lambda)_n}{(\lambda)_{2n}}$. The solution of (62) is

(i) for n even

$$a_i(n) = \frac{(-1)^i 2^i (2n)!\,\Gamma(i+\lambda)}{\Gamma(2i+\lambda)} \sum_{j=0}^{n} \frac{(-1)^j 2^{2j}}{(n-j)!(2j-i)!} \times {}_2F_1(-(2j-i), i+\beta+1; 2i+\lambda+1; 2)$$

(ii) for n odd

$$a_i(n) = \frac{(-1)^i 2^{i+1} (2n+1)!\,\Gamma(i+\lambda)}{\Gamma(2i+\lambda)} \sum_{j=0}^{n} \frac{(-1)^j 2^{2j}}{(n-j)!(2j-i+1)!} \times {}_2F_1(-(2j-i+1), i+\beta+1; 2i+\lambda+1; 2).$$

6. Application to ordinary differential equations with varying coefficients

Let $f(x)$ be an infinitely differentiable function defined on $[-1, 1]$, and have the Jacobi expansion (3), and assume that it satisfies the linear nonhomogeneous differential equation of order n

$$\sum_{i=0}^{n} p_i(x) f^{(i)}(x) = p(x) \tag{63}$$

where $p_0, p_1, \ldots, (p_n(x) \neq 0)$ are polynomials of x, and the coefficients of the Jacobi series of the function $p(x)$ are known, formulae (8), (17) and (32) enable one to construct in view of equation (63) the linear recurrence relation of order r, namely

$$\sum_{j=0}^{r} \alpha_j(k) a_{k+j} = \beta(k) \qquad k \geqslant 0 \tag{64}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_r$ $(\alpha_0 \neq 0, \alpha_r \neq 0)$ are polynomials of the variable k. An analytical solution of (64)—such as those given for (46), (57) and (62)—is not generally easy to obtain. The alternative approach for solving (64) can be obtained by using the well-known methods of Miller and Oliver as well as modifications and generalizations of these methods (see Jirari (1995), Luke (1969), Oliver (1969), Scraton (1972), Wimp (1984), Weixlbaumer (2001)).

Remark 4. It is of fundamental importance to note that the recurrence relations (46), (57) and (62) are minimal (i.e. the shortest in order) for the connection coefficients in (42), (56) and (61). This minimality is concluded in the connection problems considered just because they coincide with those given in Godoy *et al* (1997), displayed in table 1, p 263.

Remark 5. It should be mentioned that our goal here is to emphasize the systematic character and simplicity of our algorithm, which allows one to implement it in any computer algebra (here the Mathematica (1999) symbolic language has been used).

To end this paper, we wish to report that this work deals with formulae associated with the Jacobi coefficients for the moments of a general-order derivative of differentiable functions and with the connection coefficients between Jacobi–Jacobi and ultraspherical–ultraspherical and other combinations with different parameters. These formulae can be used to facilitate greatly the setting up of algebraic systems to be obtained by applying the spectral methods for solving differential equations with polynomial coefficients of any order, which we hope to report in a forthcoming paper.

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